Revenue Maximization for Communication Networks with Usage-Based Pricing

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***Abstract*—We study the optimal usage-based pricing problem in a resource-constrained network with one profit-maximizing Service Provider (SP) and multiple groups of surplus-maximizing users. With the assumption that the SP knows the utility function of each user (thus complete information), we find that the complete price differentiation scheme can achieve a large revenue gain (e.g., 50%) compared to no price differentiation, when the total network resource is comparably limited and the high willingness to pay users are minorities. However, the complete price differentiation scheme may lead to a high implementational complexity. To trade off the revenue against the implementational complexity, we further study the partial price differentiation scheme, and design a polynomial-time algorithm that can compute the optimal partial differentiation prices. We also consider the incomplete information case where the SP does not know which group each user belongs to. We show that it is still possible to realize price differentiation under this scenario, and provide the sufficient and necessary condition under which an incentive compatible differentiation scheme can achieve the same revenue as under complete information.**

I. Introduction

Pricing is important for the design, operation, and management of communication networks. Pricing has been used with two different meanings in the area of communication networks. One is the "optimization-oriented" pricing for network resource allocation: it is made popular by Kelly's seminal work on network congestion control [1], [2]. For example, the Transmission Control Protocol (TCP) has been successfully reverse-engineered as a congestion pricing based solution to a network optimization problem [3], [4]. A more general framework of Network Utility Maximization (NUM) was subsequently developed to forward-engineer many new network protocols (see a recent survey in [5]). In various NUM formulations, the "optimization-oriented" prices often represent the Lagrangian multipliers of various resource constraints and are used to coordinate different network entities to achieve the maximum system performance in a distributed fashion. The other is the "economics-based" pricing, which is used by a network service provider (SP) to various objectives including revenue maximization. The proper design of such a pricing becomes particularly challenging today due to the exponential growth of data volume and applications in both wireline and wireless networks. In this paper, we focus on studying the "economics-based" pricing schemes for managing communication networks.

Economists have proposed many sophisticated pricing mechanisms to extract surpluses from the consumers and maximize revenue (or profits) for the providers. A typical example is the optimal nonlinear pricing [6], [7]. In practice, however, we often observe simple pricing schemes deployed by the SPs. Typical examples include flat-fee pricing and (piecewise) linear usage-based pricing. One potential reason behind the gap between "theory" and "practice" is that the optimal pricing schemes derived in economics often has a high implementational complexity. Besides a higher maintenance cost, complex pricing schemes are not "customer-friendly" and discourage customers from using the services [8]. Furthermore, achieving the highest possible revenue often with complicated pricing schemes requires knowing the information (identity and preference) of each customer, which can be challenging in large scale communication networks. It is then natural to ask the following two questions:

1) How to design simple pricing schemes to achieve the best tradeoff between complexity and performance?
2) How does the network information structure impact the design of pricing schemes?

This paper try to answer the above two questions with some stylized communication network models. Different from many previous work in network economics that studied revenue management with the flat-fee pricing scheme [8]–[10], here we study revenue maximization with linear usage-based pricing schemes. Apparently both pricing schemes have their pros and cons. In the wireless communication world, however, the usage-based pricing scheme seems to gain increasingly more attentions due to the explosion of wireless data traffic. In June 2010, AT&T in US switched from the flat-free based pricing (unlimited data for a fixed fee) to the usage-based pricing schemes in its 3G systems [11]. Similar usage-based pricing plans have been adopted by major Chinese wireless service providers such as China Mobile and China UniCom.[1] Thus, the research on the usage-based pricing is of great practical importance.

In this paper, we consider the revenue maximization problem of a monopolist SP facing multiple groups of users. Each user determines its optimal resource demand to maximize the surplus, which is the difference between its utility and payment. The SP chooses the pricing schemes to maximize

[1] Many practical usage-based pricing plans include a fixed charge for limited amount data and then a linear usage based component for the extra date usage. For example, the AT&T iPad pricing plan charges $25 for the first 2G data, and $10 for each additional 1G data. This two-part tariff plan is complicated to analyze and is part of our future study plan.

his revenue, subject to a limited resource. We consider both complete information and incomplete information scenarios and design different pricing schemes with different implementational complexity levels. Our main contributions are as follows.

- *Complete network information*: We propose a polynomial algorithm to compute the optimal solution of the partial price differentiation problem, which includes the complete price differentiation scheme and the single pricing scheme as special cases. The optimal solution has a threshold structure, which allocates positive resources to high willingness to pay users with priorities.
- *Revenue gain under the complete network information*: Compared to the single pricing scheme, we identify the two important factors behind the revenue increase of the (complete and partial) price differentiation schemes: the differentiation gain and the effective market size. The revenue gain is the most significant when high users are minority among the whole population and total resource is limited but not too small.
- *Incomplete network information*: We design an incentive-compatible scheme with the goal to achieve the same maximum revenue that can be achieved with the complete information. We find that if the differences of willingness to pays of users are larger than some thresholds, this incentive-compatible scheme can achieve the same maximum revenue. We further characterize the necessary and sufficient condition for the thresholds.

It is interesting to compare our results under the complete network information with the one in [8], where the authors showed that the revenue gain of price differentiation is small in the flat entry-fee based Paris Metro Pricing [12]. One of the main conclusions of [8] is that a complicated differentiation strategy may not be worthwhile in a practical entry fee based pricing system. In contrast, our study here shows that the revenue gain of price differentiation can be substantial for a usage-based pricing system.

Some recent work of usage-based pricing and revenue management in communication network includes [13]–[20]. Basar and Srikant in [13] investigated the bandwidth allocation problem in a single link network with the single pricing scheme. Shen and Basar in [14] extended the study to a more general nonlinear pricing case with the incomplete network information scenario. They discussed the single pricing scheme under incomplete information with a continuum distribution of users' types. In contrast, our study on the incomplete information focuses on the linear pricing with a discrete setting of users' types. We also show that, besides the single pricing scheme, it is also possible to design differentiation pricing schemes under incomplete information. Daoud *et al.* [15] studied a uplink power allocation problem in a CDMA system, where the interference among users are the key constraint instead of the limited total resource considered in our paper. Jiang et al. in [16] and Hande et al. in [17] focused on the study of the time-dependent pricing. He and Walrand in [18], Shakkottai and Srikant in [19] and Gajic et al. in [20] focused on the interaction between different service providers embodied in the pricing strategies, rather than the design of the pricing mechanism. Besides, none of the related work considered the partial differential pricing as in our paper.

II. System Model

We consider a network with a total amount of S limited resource (which can be in the form of rate, bandwidth, power, time slot, etc.). The resource is allocated by a monopolistic SP to a set $\mathcal{I} = \{1, \ldots, I\}$ of user groups. Each group $i \in \mathcal{I}$ has N_i homogeneous users[2] with the same utility function:

$$u_i(s_i) = \theta_i \ln(1 + s_i), \tag{1}$$

where s_i is the allocated resource to one user in group i and θ_i represents the willingness to pay of group i. The logarithmic utility function here is a common model in many literatures (see [13] Remark II-1 for detailed explanations). The analysis in this paper can also be partially extended to general utility functions, with details found in our online technical report [21]. Without loss of generality, we assume that $\theta_1 > \theta_2 > \cdots > \theta_I$. In other words, group 1 contains users with the highest valuation, and group I contains users with the lowest valuation.

We consider two types of information structures:

1) **Complete information**: the SP knows each user's utility function. Though the complete information is a very strong assumption, it is the most frequently studied scenario the network pricing literature [13]–[15], [20]. The significance of studying the complete information is two-fold. It serves as the benchmark of practical designs and provides important insights for the incomplete information analysis.
2) **Incomplete information**: the SP knows the total number of groups I, the number of users in each group $N_i, i \in \mathcal{I}$, and the utility function of each group $u_i, i \in \mathcal{I}$. It does not know which user belongs to which group. Such assumption in our discrete setting is analogous to that the SP knows only the users' types distribution in a continuum case. Such statistical information can be obtained through long term observations of a stationary user population.

The interaction between the SP and users can be characterized as a two-stage Stackelberg model shown in Fig. 1. The

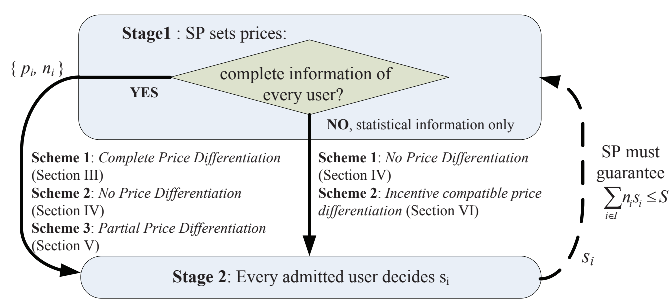

Fig. 1. A two-stage leader and follower model

SP publishes the pricing scheme in Stage 1, and users respond with their demands in Stage 2. The users want to maximize their surpluses by optimizing their demands according to the

[2] A special case is N_i=1 for each group, i.e., all users in the network are different.

pricing scheme. The SP wants to maximize its revenue by setting the right pricing scheme to induce desirable demands from users. Since the SP has a limited total resource, he must guarantee that the total demand from users is no larger than what he can supply.

The details of pricing schemes depend on the information structure of the SP. Under complete information, since the SP can distinguish different groups of users, he announces the pricing and the admission control decisions to different groups of users. It can choose from the complete price differentiation scheme, the single price scheme, and the partial price differentiation scheme to realize desired trade-off between implementational complexity and total revenue. Under incomplete information, it publishes a common price *menu* to all users, and allow users to freely choose a particular price option in this menu. All these pricing schemes will be discussed one by one in the following sections.

III. Complete Price Differentiation under complete information

We first consider the complete information case. Since the SP knows the utility and identity of each user, it is possible to maximize the revenue by charging a different price to each group of users. The analysis will be based on backward induction, starting from Stage 2 and then moving to Stage 1.

A. User's Surplus Maximization Problem in Stage 2

If a user in group i has been admitted into the network and offered a linear price p_i in Stage 1, then it solves the following surplus maximization problem,

$$\max_{s_i \geq 0} \; u_i(s_i) - p_i s_i, \tag{2}$$

which leads to the following unique optimal demand

$$s_i(p_i) = \left(\frac{\theta_i}{p_i} - 1\right)^+, \text{ where } (\cdot)^+ \triangleq \max(\cdot, 0). \tag{3}$$

Remark 1: The analysis of the Stage 2 user surplus maximization problem is the same for all pricing schemes. The result in (3) will be also used in Sections IV, V and VI.

B. SP's Pricing and Admission Control Problem in Stage 1

In Stage 1, the SP maximizes its revenue by choosing the price p_i and the admitted user number n_i for each group i subject to the limited total resource S. The key idea is to perform a Complete Price differentiation (CP) scheme, i.e., charging each group with a different price.

$$CP: \quad \underset{\boldsymbol{p}\geq 0, \boldsymbol{s}\geq 0, \boldsymbol{n}}{\text{maximize}} \quad \sum_{i\in\mathcal{I}} n_i p_i s_i \tag{4}$$

$$\text{subject to} \quad s_i = \left(\frac{\theta_i}{p_i} - 1\right)^+, \; i \in \mathcal{I}, \tag{5}$$

$$n_i \in \{0, \ldots, N_i\} \;, \; i \in \mathcal{I}, \tag{6}$$

$$\sum_{i\in\mathcal{I}} n_i s_i \leq S. \tag{7}$$

where $\boldsymbol{p} \triangleq \{p_i, i \in \mathcal{I}\}$, $\boldsymbol{s} \triangleq \{s_i, i \in \mathcal{I}\}$, and $\boldsymbol{n} \triangleq \{n_i, i \in \mathcal{I}\}$. We use bold symbols to denote vectors in the sequel. Constraint (5) is the solution of the Stage 2 user surplus maximization problem in (3). Constraint (6) denotes the admission control decision, and constraint (7) represents the total limited resource in the network.

CP Problem is not straightforward to solve, since it is a non-convex optimization problem with a non-convex objective function (summation of products of n_i and p_i), a coupled constraint (7), and integer variables $\boldsymbol{n}$. However, it is possible to convert it into an equivalent convex formulation through a series of transformations, and thus the problem can be solved efficiently.

First, we can remove the $(\cdot)^+$ sign in constraint (5) by realizing the fact that there is no need to set p_i higher than θ_i for users in group i; users in group i already demand zero resource and generate zero revenue when $p_i = \theta_i$. This means that we can rewrite constraint (5) as

$$p_i = \frac{\theta_i}{s_i + 1} \text{ and } s_i \geq 0, i \in \mathcal{I}. \tag{8}$$

Plugging (8) into (4), then the objective function becomes $\sum_{i\in\mathcal{I}} n_i \frac{\theta_i s_i}{s_i+1}$. We can further decompose the CP Problem in the following two sub-problems:

1) *Resource allocation*: for a fixed admission control decision $\boldsymbol{n}$, solve for the optimal resource allocation $\boldsymbol{s}$.

$$CP_1: \quad \underset{\boldsymbol{s}\geq 0}{\text{maximize}} \quad \sum_{i\in\mathcal{I}} n_i \frac{\theta_i s_i}{s_i + 1}$$
$$\text{subject to} \quad \sum_{i\in\mathcal{I}} n_i s_i \leq S. \tag{9}$$

Denote the solution of CP_1 as $\boldsymbol{s}^* = (s_i^*(\boldsymbol{n}), \forall i \in \mathcal{I})$. We further maximize the revenue of the integer admission control variables $\boldsymbol{n}$.

2) *Admission control problem*:

$$CP_2: \quad \underset{\boldsymbol{n}}{\text{maximize}} \quad \sum_{i\in\mathcal{I}} n_i \frac{\theta_i s_i^*(\boldsymbol{n})}{s_i^*(\boldsymbol{n}) + 1} \tag{10}$$
$$\text{subject to} \quad n_i \in \{0, \ldots, N_i\} \;, \; i \in \mathcal{I}$$

Let us first solve CP_1 subproblem in $\boldsymbol{s}$. Note that it is a convex optimization problem. By using Lagrange multiplier technique, we can get the first-order necessary and sufficient condition:

$$s_i^*(\lambda) = \left(\sqrt{\frac{\theta_i}{\lambda}} - 1\right)^+, \tag{11}$$

where λ be the Lagrange multiplier corresponding to the resource constraint (9).

Meanwhile, we note the resource constraint (9) must hold with equality, since the objective is strictly increasing function in s_i. Thus, by plugging (11) into (9), we have

$$\sum_{i\in\mathcal{I}} n_i \left(\sqrt{\frac{\theta_i}{\lambda}} - 1\right)^+ = S. \tag{12}$$

This weighted water-filling problem (where $\frac{1}{\sqrt{\lambda}}$ can be viewed as the water level) in general has no closed-form solution for λ. However, we can efficiently determine the optimal solution λ^* by exploiting the special structure of our problem. Note that since $\theta_1 > \cdots > \theta_I$, then λ^* must satisfy the following condition:

$$\sum_{i=1}^{K^{cp}} n_i \left(\sqrt{\frac{\theta_i}{\lambda^*}} - 1\right) = S, \tag{13}$$

for a group index threshold value K^{cp} satisfying

$$\frac{\theta_{K^{cp}}}{\lambda^*} > 1 \text{ and } \frac{\theta_{K^{cp}+1}}{\lambda^*} \leq 1. \tag{14}$$

In other words, only groups with index no larger than K_{cp} will be allocated the positive resource. This property leads to the following simple Algorithm 1 to compute λ^* and group index threshold K^{cp}: we start by assuming $K^{cp} = I$ and compute λ. If (14) is not satisfied, we decrease K^{cp} by one and recompute λ until (14) is satisfied.

Algorithm 1 Solving the Resource Allocation Problem CP_1

function $CP(\{n_i, \theta_i\}_{i\in\mathcal{I}}, S)$
  $k \leftarrow I$, $\lambda(k) \leftarrow \left(\frac{\sum_{i=1}^{k} n_i\sqrt{\theta_i}}{S+\sum_{i=1}^{k} n_i}\right)^2$
  while $\theta_k \leq \lambda(k)$ **do**
    $k \leftarrow k-1$, $\lambda(k) \leftarrow \left(\frac{\sum_{i=1}^{k} n_i\sqrt{\theta_i}}{S+\sum_{i=1}^{k} n_i}\right)^2$
  end while
  $K^{cp} \leftarrow k$, $\lambda^* \leftarrow \lambda(k)$
  return K^{cp}, λ^*
end function

Since $\theta_1 > \lambda(1) = (\frac{n_1}{s+n_1})^2\theta_1$, Algorithm 1 always converges and returns the unique values of K^{cp} and λ^*. The complexity is $\mathcal{O}(I)$, i.e., linear in the number of user groups (not the number of users).

With K^{cp} and λ^*, the solution of the resource allocation problem can be written as

$$s_i^* = \begin{cases} \sqrt{\frac{\theta_i}{\lambda^*}} - 1, & i = 1, \ldots, K^{cp}; \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

For the ease of discussions, we introduce a new notion of the *effective market*, which denotes all the groups allocated non-zero resource. For resource allocation problem CP_1, the threshold K^{cp} describes the size of the effective market. All groups with indices no larger than K^{cp} are *effective group*s, and users in these groups as *effective user*s. An example of the effective market is illustrated in Fig. 2.

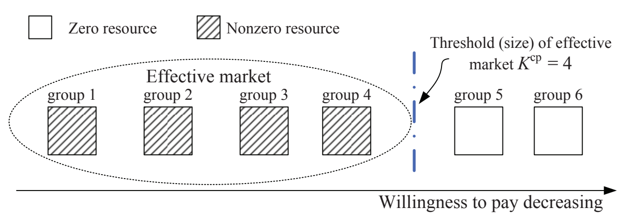

Fig. 2. A 6-group example for effective market: the willingness to pays decrease from group 1 to group 6. The effective market threshold can be obtained by Algorithm 1, and is 4 in this example.

Now let us solve the admission control problem CP_2. Denote the objective (10) as $R_{cp}(\boldsymbol{n})$, by (15), then $R_{cp}(\boldsymbol{n}) \triangleq \sum_{i=1}^{K^{cp}} n_i \left(\sqrt{\frac{\theta_i}{\lambda^*(\boldsymbol{n})}} - 1\right)\sqrt{\theta_i\lambda^*(\boldsymbol{n})}$. We first relax the integer domain constraint of n_i as $n_i \in [0, N_i]$. Since (13), by taking the derivative of the objective function $R_{cp}(\boldsymbol{n})$ with respect to n_i, we have

$$\frac{\partial R_{cp}(\boldsymbol{n})}{\partial n_i} = \left(\sum_{i=1}^{K^{cp}} n_i \left(\sqrt{\frac{\theta_i}{\lambda^*(\boldsymbol{n})}} - 1\right)\right)\frac{\partial\sqrt{\theta_i\lambda^*(\boldsymbol{n})}}{\partial n_i}, \tag{16}$$

Also from (13), we have $\lambda^* = \left(\frac{\sum_{i=i}^{K^{cp}} n_i\sqrt{\theta_i}}{S+\sum_{i=1}^{K^{cp}} n_i}\right)^2$, thus $\frac{\partial\sqrt{\lambda^*(\boldsymbol{n})}}{\partial n_i} > 0$, for $i = 1, \ldots, K^{cp}$, and $\frac{\partial\sqrt{\lambda^*(\boldsymbol{n})}}{\partial n_i} = 0$, for $i = K^{cp}+1, \ldots, I$. This means that the objective $R_{cp}(\boldsymbol{n})$ is strictly increasing in n_i for all $i = 1, \ldots, K^{cp}$, thus it is optimal to admit all users in the effective market. The admission decisions for the groups not in the effective market is irrelevant to the optimization, since those users consume zero resource. Therefore, one of the optimal solutions of CP_1^2 subproblem is $n_i^* = N_i$ for all $i \in \mathcal{I}$.

Solving CP_1 and CP_2 subproblems leads to the optimal solution of CP Problem:

Theorem 1: There exists an optimal solution of CP Problem that satisfies the following conditions:

- All users are admitted: $n_i^* = N_i$ for all $i \in \mathcal{I}$.
- There exist a value λ^* and a group index threshold $K^{cp} \leq I$, such that only the top K^{cp} groups of users receive positive resource allocations,

$$s_i^* = \begin{cases} \sqrt{\frac{\theta_i}{\lambda^*}} - 1, & i = 1, \ldots, K^{cp}; \\ 0, & \text{otherwise.} \end{cases}$$

with the prices

$$p_i^* = \begin{cases} \sqrt{\theta_i\lambda^*}, & i = 1, \ldots, K^{cp}; \\ \theta_i, & \text{otherwise.} \end{cases}$$

The values of λ^* and K^{cp} can be computed as in Algorithm 1 by setting $n_i = N_i$, for all $i \in \mathcal{I}$.

Theorem 1 provides the right economic intuition: SP maximizes its revenue by charging a higher price to users with a higher willingness to pay. It is easy to check that $p_i > p_j$ for any $i < j$. The small willingness to pay users are excluded from the markets.

C. Properties

Here we summarize some interesting properties of the optimal complete price differentiation scheme:

1) Threshold structure: The threshold based resource allocation means that higher willingness to pay groups have higher priories of obtaining the resource at the optimal solution.

To see this clearly, assume the effective market size is $K^{(1)}$ under parameters $\{\theta_i, N_i^{(1)}\}_{i\in\mathcal{I}}$ and S. Here the superscript (1) denotes the first parameter set. Now consider another set of parameters $\{\theta_i, N_i^{(2)}\}_{i\in\mathcal{I}}$ and S, where $N_i^{(2)} \geq N_i^{(1)}$ for each group i and the new The effective market size is $K^{(2)}$. By (13), we can see that $K^{(2)} \leq K^{(1)}$. Furthermore, we can show that if some high willingness to pay group has many more users under the latter system parameters, i.e., $N_i^{(2)}$ is much larger than $N_i^{(1)}$ for some $i < K^{(1)}$, then the effective size will be strictly decreased, i.e., $K^{(2)} < K^{(1)}$. In other words, the increase of high willingness to pay users will drive the low willingness to pay users out of the effective market.

2) Admission control with pricing: Theorem 1 shows the explicit admission control is not necessary at the optimal solution. Also from Theorem 1, we can see that when the number of users in any effective group increases, the price p_i^*, for all $i \in \mathcal{I}$ increases and resource s_i^*, for all $\forall\, i \leq K^{cp}$ decreases. The prices serve as the indications of the scarcity of the resources and will automatically prevent the low willingness to pay users to consume the network resource.

IV. Single Pricing Scheme

In last section, we show that the CP scheme is the optimal pricing scheme to maximize the revenue under complete information. However, such a complicated pricing scheme is of high implementational complexity. In this section we study the single pricing scheme. It is clear that the scheme will in general suffer a revenue loss comparing with the CP scheme. We will try to characterize the impact of various system parameters on such revenue loss.

A. Problem formulation and solution

Let us first formulate the Single Pricing (SP) problem.

$$\begin{aligned} SP: \quad & \underset{p\geq 0,\ \boldsymbol{n}}{\text{maximize}} && p\sum_{i\in\mathcal{I}} n_i s_i \\ & \text{subject to} && s_i = \left(\frac{\theta_i}{p}-1\right)^+, \ \ i\in\mathcal{I} \\ & && n_i \in \{0,\ldots,N_i\} \ \ , \ \ i\in\mathcal{I} \\ & && \sum_{i\in\mathcal{I}} n_i s_i \leq S. \end{aligned}$$

Comparing with CP Problem in Section III, here the SP charges a single price p to all groups of users. After a similar transformation as in Section III, we can show that the optimal single price satisfies the following the weighted water-filling condition

$$\sum_{i\in\mathcal{I}} N_i \left(\frac{\theta_i}{p}-1\right)^+ = S.$$

Thus we can obtain the following solution that shares a similar structure as complete price differentiation.

Theorem 2: There exists an optimal solution of SP Problem that satisfies the following conditions:

- All users are admitted: $n_i^* = N_i$, for all $i\in\mathcal{I}$.
- There exist a price p^* and a group index threshold $K^{sp}\leq I$, such that only the top K^{sp} groups of users receive positive resource allocations,

$$s_i^* = \begin{cases} \frac{\theta_i}{p^*}-1, & i=1,2,\ldots,K^{sp}, \\ 0, & \text{otherwise}, \end{cases}$$

with the price

$$p^* = p(K^{sp}) = \frac{\sum_{i=1}^{K^{sp}} N_i\theta_i}{S+\sum_{i=1}^{K^{sp}} N_i}.$$

The value of K^{sp} and p^* can be computed as in Algorithm 2.

Algorithm 2 Search the threshold of the SP Problem

function $SP(\{N_i,\theta_i\}_{i\in\mathcal{I}},\ S)$
  $k \leftarrow I,\ p(k) \leftarrow \frac{\sum_{i=1}^{k} N_i\theta_i}{S+\sum_{i=1}^{k} N_i}$
  while $\theta_k \leq p(k)$ **do**
    $k \leftarrow k-1,\ p(k) \leftarrow \frac{\sum_{i=1}^{k} N_i\theta_i}{S+\sum_{i=1}^{k} N_i}$
  end while
  $K^{sp} \leftarrow k,\ p^* \leftarrow p(k)$
  return $K^{sp},\ p^*$
end function

B. Properties

The SP scheme shares several similar properties as the CP scheme (Sec. III-C), including the threshold structure and admission control with pricing. Similarly, we can define the effective market for the SP scheme.

It is more interesting to notice the differences between these two schemes. To distinguish solutions, we use the superscript "cp" for the CP scheme, and "sp" for the SP scheme.

Proposition 1: Under same parameters $\{N_i,\theta_i\}_{i\in\mathcal{I}}$ and S:

1) The effective market of the SP scheme is no larger than the one of the CP scheme, i.e., $K^{sp}\leq K^{cp}$.
2) There exists a threshold $\bar{k}\in\{1,2\ldots,K^{sp}\}$, such that
   - Groups with indices less than $\bar{k}$ (low willingness to pay users) are charged with higher prices and allocated less resources in the SP scheme, i.e., $s_i^{cd}\leq s_i^{sp}$ and $p_i^{cd}\geq p^*$, $\forall\, i\leq\bar{k}$, where the equality holds if only if $i=\bar{k}$ and $\theta_{\bar{k}} = \frac{p^{*2}}{\lambda^*}$,
   - Groups with indices greater than $\bar{k}$ (high willingness to pay users) are charged with lower prices and allocated more resources in the SP scheme, i.e., $s_i^{cd} > s_i^{sp}$ and $p_i^{cd} < p^*$, $\forall\, i\geq\bar{k}$.

   where p^* is the optimal single price.

The proof is given in Appendix-A. An illustrative example is shown in Fig. 3 and Fig. 4.

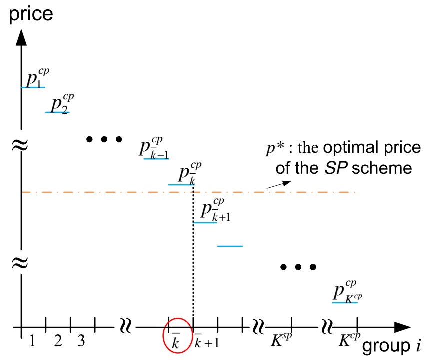

Fig. 3. Comparison of prices between the CP scheme and the SP scheme

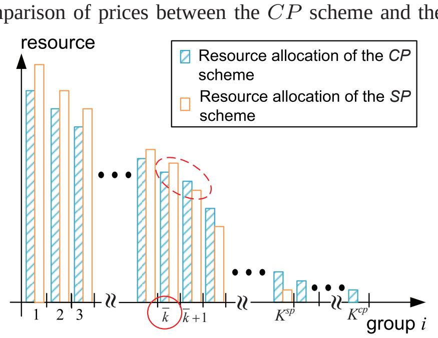

Fig. 4. Comparison of resource allocation between the CP scheme and the SP scheme

It is easy to understand that the SP scheme makes less revenue, since it is a feasible solution to CP Problem. A little bit more computation sheds more light on this comparison. We introduce the following notations to streamline the comparison:

- $N_{eff}(k) \triangleq \sum_{i=1}^{k} N_i$: the number of effective users, where k is the size of the effective market.

- $\gamma_i(k) \triangleq \frac{N_i}{N_{eff}(k)}$, $i = 1, 2, \ldots, k$: the fraction of group i's users in the effective market.
- $\bar{s}(k) \triangleq \frac{S}{N_{eff}(k)}$: the average resource per an effective user.
- $\bar{\theta}(k) \triangleq \sum_{i=1}^{k} \gamma_i \theta_i$: the average willingness to pay per an effective user.

Based on Theorem 1, the revenue of the CP scheme is

$$R^{cp}(K^{cp}) = N_{eff}(K^{cp}) \left(\frac{\bar{s}(K^{cp})\bar{\theta}(K^{cp}) + g(K^{cp})}{\bar{s}(K^{cp}) + 1} \right), \quad (17)$$

where

$$g(K^{cp}) = \frac{1}{\lambda^*} \sum_{i=1}^{K^{cp}} \sum_{j>i}^{K^{cp}} \gamma_i \gamma_j (p_i^{cp} - p_j^{cp})^2. \quad (18)$$

Based on Theorem 2, the revenue of the SP scheme is

$$R^{sp}(K^{sp}) = N_{eff}(K^{sp}) \left(\frac{\bar{s}(K^{sp})\bar{\theta}(K^{sp})}{\bar{s}(K^{sp}) + 1} \right). \quad (19)$$

From (17) and (19), it is clear to see that $R^{cp} \geq R^{sp}$ due to two factors: one is the non-negative term in (18), the other is $K^{cp} \geq K^{sp}$: a higher level of differentiation implies a no smaller effective market. Let us further discuss them in the following two cases:

- If $K^{cp} = K^{sp}$, then the additional term of (18) in (17) means that $R^{cp} \geq R^{sp}$. The equality holds if and only if $K^{cp} = 1$, in which case $g(K^{cp}) = 0$. Note that in this case, the CP scheme degenerates to the SP scheme. We name the nonnegative term $g(K^{cp})$ in (18) as *price differentiation gain*, as it measures the average price difference between any effective users in the CP scheme. The larger the price difference, the larger the gain. When there is no differentiation in the degenerating case ($K^{cp} = 1$), the gain is zero.
- If $K^{cp} > K^{sp}$, since the common part of two revenue $N_{eff}(K) \left(\frac{\bar{s}(K)\bar{\theta}(K)}{\bar{s}(K)+1} \right) = \frac{S\bar{\theta}N_{eff}(K)}{S+N_{eff}(K)}$ is a strictly increasing function of $N_{eff}(K)$, price differentiation makes more revenue even if the positive differentiation gain $g(K^{cp})$ is not taken into consideration. This result is intuitive, that more consumers with purchasing power always mean more revenue in the service provider's pocket.

Finally, we note that the CP scheme in Section III requires the complete network information. The SP scheme here, on the other hand, works in the incomplete information case as well. This distinction becomes important in Section VI.

V. Partial Price Differentiation under Complete Information

As we discussed in last section, the complete price differentiation scheme may be too complicated to implement. However, the single pricing scheme may suffer a considerable revenue loss. How to achieve a good tradeoff between the implementational complexity and the total revenue? In reality, we usually see that the SP provide only a few pricing plans for the entire users population; we term it as the *partial price differentiation* scheme. In this section, we will answer the following question: if the SP is constrained to maintain a limited number of prices, $p^1, \ldots, p^J$, $J \leq I$, then what is the optimal pricing strategy and the maximum revenue? Concretely, the Partial Price differentiation (PP) problem is formulated as follows.

$$\begin{aligned}
PP: \quad & \underset{n_i, p_i, s_i, p^j, a_i^j}{\text{maximize}} && \sum_{i \in \mathcal{I}} n_i p_i s_i \\
& \text{subject to} && s_i = \left(\frac{\theta_i}{p_i} - 1 \right)^+, \; \forall\, i \in \mathcal{I}, && (20) \\
& && n_i \in \{0, \ldots, N_i\}, \; \forall\, i \in \mathcal{I}, && (21) \\
& && \sum_{i \in \mathcal{I}} n_i s_i \leq S, && (22) \\
& && p_i = \sum_{j \in \mathcal{J}} a_i^j p^j, && (23) \\
& && \sum_{j \in \mathcal{J}} a_i^j = 1, \; a_i^j \in \{0, 1\}, \; \forall\, i \in \mathcal{I}.
\end{aligned}$$

Here $\mathcal{J}$ denotes the set $\{1, 2, \ldots, J\}$. Compared to CP and SP Problems, there is one more constraint (23) here, which means that p_i charged to each group i is chosen from a set of J prices $\{p^j, j \in \mathcal{J}\}$. For convenience, we define *cluster* $\mathcal{C}^j \triangleq \{i \,|\, a_i^j = 1\}$, $j \in \mathcal{J}$, which is a set of groups charged with the same price p^j. We use superscript j to denote clusters, and subscript i to denote groups through this section. We term the binary variables $\boldsymbol{a} \triangleq \{a_i^j, \; j \in \mathcal{J}, \; i \in \mathcal{I}\}$ as the *partition*, which determines which cluster each group belongs to.

PP Problem is a combinatorial optimization problem, and is more difficult than the previous CP and SP Problems. On the other hand, we notice that this PP Problem formulation includes the CP scheme ($J = I$) and the SP scheme scenario ($J = 1$) as special cases. The insights we obtained from solving these two special cases in Sections III and IV will be helpful to solve the general PP problem.

A. Three-level Decomposition

To solve PP Problem, we decompose and tackle it in three levels. In the lowest level-3, we determine the pricing and resource allocation for each cluster, given a fixed partition and fixed resource allocation among clusters. In level-2, we compute the optimal resource allocation among clusters, given a fixed partition. In level-1, we optimize the partition among groups.

1) Level-3: Pricing and resource allocation in each cluster: For a fix partition $\boldsymbol{a}$ and a cluster resource allocation $\boldsymbol{s} \triangleq \{s^j\}_{j \in \mathcal{J}}$, we focus the pricing and resource allocation problems within each cluster $\mathcal{C}^j$, $j \in \mathcal{J}$:

$$\begin{aligned}
\text{Level-3:} \quad & \underset{n_i, s_i, p^j}{\text{maximize}} && \sum_{i \in C^j} n_i p^j s_i \\
& \text{subject to} && s_i = \left(\frac{\theta_i}{p^j} - 1 \right)^+, \quad \forall\, i \in \mathcal{C}^j, \\
& && n_i \leq N_i, \quad \forall\, i \in \mathcal{C}^j, \\
& && \sum_{i \in \mathcal{C}^j} n_i s_i \leq s^j.
\end{aligned}$$

This Level-3 subproblem coincides with the SP scheme discussed in Section IV, since all groups within the same cluster $\mathcal{C}^j$ are charged with a single price p^j. We can then

directly apply the results in Theorem 2 to solve the Level-3 problem. We denote the effective market threshold[3] for cluster $\mathcal{C}^j$ as K^j, which can be computed in Algorithm 2. An illustrative example is shown in Fig. 5, where the cluster contains four groups (group 4, 5, 6 and 7), and the effective market contains groups 4 and 5, thus $K^j = 5$. The SP obtains the following maximum revenue obtained from cluster $\mathcal{C}^j$:

$$R^j(s^j, \boldsymbol{a}) = \frac{s^j \sum_{i \in \mathcal{C}^j,\, i \le K^j} N_i \theta_i}{s^j + \sum_{i \in \mathcal{C}^j,\, i \le K^j} N_i}. \tag{24}$$

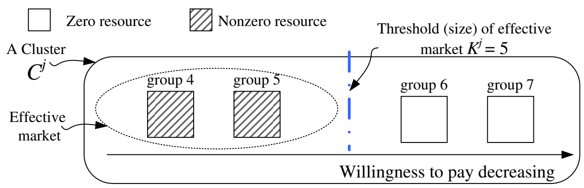

Fig. 5. An illustrative example: the cluster contains four groups, group 4, 5, 6 and 7; and the effective market contains group 4 and 5, thus $K^j = 5$

2) Level-2: Resource allocation among clusters: For a fix partition $\boldsymbol{a}$, we then consider the resource allocation among clusters.

$$\begin{aligned} \text{Level-2:} \quad & \underset{s^j \ge 0}{\text{maximize}} \quad \sum_{j \in \mathcal{J}} R^j(s^j, \boldsymbol{a}) \\ & \text{subject to} \quad \sum_{j \in \mathcal{J}} s^j \le S \end{aligned}$$

We will show in Section V-B that subproblems in Level-2 and Level-3 can be transformed into a complete price differentiation problem under proper technique conditions. Let us denote the its optimal value as $R_{pp}(\boldsymbol{a})$.

3) Level-1: cluster partition: Finally, we solve the cluster partition problem.

$$\begin{aligned} \text{Level-1:} \quad & \underset{a_i^j \in \{0,1\}}{\text{maximize}} \quad R_{pp}(\boldsymbol{a}) \\ & \text{subject to} \quad \sum_{j \in \mathcal{J}} a_i^j = 1,\ i \in \mathcal{I}. \end{aligned}$$

This partition problem is a combinatorial optimization problem. The size of its feasible set is $S(I, J) = \frac{1}{J!} \sum_{t=1}^{J} (-1)^{J+t} C(J,t) t^I$, *Stirling number of the second kind* [22, Chap.13], where $C(J,t)$ is the binomial coefficient. Some numerical examples are given in the third row in Table I. If the number of prices J is given, the feasible set size is exponential in the total number of groups I. For our problem, however, it is possible to reduce the size of the feasible set by exploiting the special problem structure. More specifically, the group indices in each cluster should be consecutive at the optimum. This means that the size of the feasible set is $C(I-1, J-1)$ as shown in the last row in Table I, and thus is much smaller than $S(I, J)$.

Next we discuss how to solve the three level subproblems. A route map for the whole solving process is given in Fig. 6.

[3]Note that we do not assume that the effective market threshold equals to the number of effective groups, e.g., there are 2 effective groups in Fig. 5, but threshold $K^j = 5$. Later we will prove that there is unified threshold for PP Problem. Then by this result, the group index threshold actually coincides with the number of effective groups.

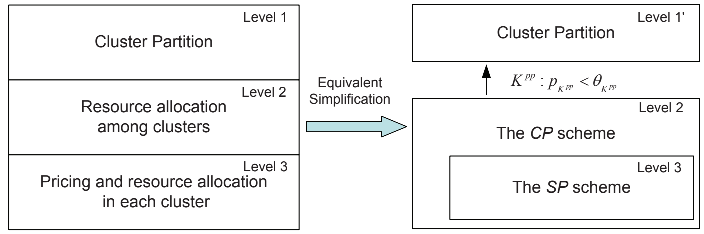

Fig. 6. Decomposition and simplification of the general PP Problem: The three-level decomposition structure of PP Problem is shown in the left hand side. After simplifications in Section V-B and V-C, the problem will be reduced to structure in right hand side.

B. Solving Level-2 and Level-3

The optimal solution (24) of the Level-3 problem can be equivalently written as

$$R^j(\boldsymbol{s}, \boldsymbol{a}) = \frac{s^j \sum_{i \in \mathcal{C}^j,\, i \le K^j} N_i \theta_i}{s^j + \sum_{i \in \mathcal{C}^j,\, i \le K^j} N_i} \overset{(a)}{=} \frac{s^j N^j \theta^j}{s^j + N^j}, \tag{25}$$

$$\text{where} \begin{cases} N^j & = \sum_{i \in \mathcal{C}^j,\, i \le K^j} N_i, \\ \theta^j & = \sum_{i \in \mathcal{C}^j,\, i \le K^j} \frac{N_i \theta_i}{N^j}. \end{cases} \tag{26}$$

The equality (a) in (25) means that each cluster $\mathcal{C}^j$ can be equivalently treated as a group with N^j homogeneous users with the same willings to pay θ^j. We name this equivalent group as a *super-group* (SG). We summarize the above result as the following lemma.

Lemma 1: For every cluster C^j and total resource s^j, $j \in \mathcal{J}$, we can find an equivalent super-group which satisfies conditions in (26) and achieves the same revenue under the SP scheme.

Based on Lemma 1, Level-2 and level-3 subproblems together can be viewed as a CP problem for super-groups. Since a cluster and its super-group from a one-to-one mapping, we will use the two words interchangeably in the sequel.

However, simply combining Theorems 1 and 2 to solve Level-2 and Level-3 subproblems for a fixed partition $\boldsymbol{a}$ can result in a very high complexity. This is because the effective markets within each super-group and between super-groups are coupled together. An illustrative example of this coupling effective market is shown in Fig. 7, where K^c is the threshold between clusters and has three possible positions (i.e., between group 2 and group 3, between group 5 and group 6, or after group 6); and K_1 and K_2 are thresholds within cluster $\mathcal{C}^1$ and $\mathcal{C}^2$, which have two or three possible positions, respectively. Thus, there are $(2\times3)\times3 = 18$ possible thresholds possibilities in total.

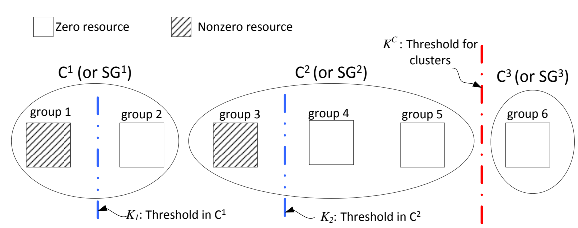

Fig. 7. An example of coupling thresholds.

The key idea to resolve this coupling issue is to show that the situation in Fig. 7 can not be an optimal solution of PP Problem. The results in Sections III and IV show that

TABLE I
NUMERICAL EXAMPLES FOR FEASIBLE SET SIZE OF THE PARTITION PROBLEM IN LEVEL-1

Number of groups	$I = 10$		$I = 100$		$I = 1000$
Number of prices	$J = 2$	$J = 3$	$J = 2$	$J = 3$	$J = 2$
$S(I, J)$	511	9330	6.33825×10^{29}	8.58963×10^{46}	5.35754×10^{300}
$C(I-1, J-1)$	9	36	99	4851	999

there is a unified threshold at the optimum in both the CP and SP cases, e.g., Fig. 2. Next we will show that a unified single threshold also exists in the PP case.

Lemma 2: At any optimal solution of the PP scheme, the group indices of the effective market is consecutive.

The proof of Lemma 2 is given in Appendix B. The intuition is that the resource should be always allocated to high willingness to pay users at the optimum. Thus, it is not possible to have Fig. 7 at an optimal solution, where high willingness to pay users in group 2 are allocated zero resource while low willingness to pay users in group 3 are allocated positive resources.

Based on Lemma 2, we know that there is a unified effective market threshold for PP Problem, denoted as K^{pp}. Since all groups with indices larger than K^{pp} make zero contribution to the revenue, we can ignore them and only consider the partition problem for the first K^{pp} groups. Given a partition that divides the K^{pp} groups into J clusters (super-groups), we can apply the CP result in Section III to compute the optimal revenue in Level-2 based on Theorem 1.

$$R_{pp}(\boldsymbol{a}) = \sum_{j=1}^{J} N^j \theta^j - \frac{\left(\sum_{j=1}^{J} N^j \sqrt{\theta^j}\right)^2}{S + \sum_{j=1}^{J} N^j}$$
$$= \sum_{i=1}^{K^{pp}} N_i \theta_i - \frac{\left(\sum_{j=1}^{J} N^j \sqrt{\theta^j}\right)^2}{S + \sum_{i=1}^{K^{pp}} N_i}. \quad (27)$$

C. Solving Level-1

1) With a given effective market threshold K^{pp}: Based on the previous results, we first simplify the Level-1 subproblem, and prove the theorem below.

Theorem 3: For a given threshold K^{pp}, the optimal partition of Level-1 subproblem is the solution of the following optimization problem.

$$\text{Level-1}' \quad \underset{a_i^j, N^j, \theta^j}{\text{minimize}} \quad \sum_{j \in \mathcal{J}} N^j \sqrt{\theta^j}$$
$$\text{subject to} \quad N^j = \sum_{i \in \mathcal{K}^{pp}} N_i a_i^j, \ \ j \in \mathcal{J},$$
$$\theta^j = \sum_{i \in \mathcal{K}^{pp}} \frac{N_i a_i^j}{N^j} \theta_i \ \ j \in \mathcal{J},$$
$$\sum_{j \in \mathcal{J}} a_i^j = 1, \ a_i^j \in \{0, 1\}, i \in \mathcal{K}^{pp} \ j \in \mathcal{J},$$
$$\theta_{K^{pp}} > p^J = \sqrt{\theta^J(\boldsymbol{a}) \lambda(\boldsymbol{a})}. \quad (28)$$

where $\mathcal{K}^{pp} \triangleq \{1, 2, \ldots, K^{pp}\}$, $\theta^J(\boldsymbol{a})$ is the value of average willingness to pay of the Jth group for the partition $\boldsymbol{a}$, and $\lambda(\boldsymbol{a}) = \left(\frac{\sum_{j \in \mathcal{J}} N^j \sqrt{\theta^j}}{S + \sum_{i=1}^{K^{pp}} N_i}\right)^2$.

Proof: The objective function and the first three constraints in Level-1′ Problem are easy to understand: if the effective market threshold K^{pp} is given, then the objective function of the Level-1 subproblem, maximizing R_{pp} in (27) over $\boldsymbol{a}$, is as simple as minimizing $\sum_{j=1}^{J} N^j \sqrt{\theta^j}$ as the Level-1′ Problem suggested; the first three constraints are given by the definition of the partition.

Constraint (28) is the threshold condition that supports (27), which means that the least willingness to pay users in the effective market has a positive demand. It ensures that calculating the revenue by (27) is valid. Remember that the solution of CP Problem of Level-2 and Level-3 is threshold based, and Lemma 2 indicates that (28) is sufficient for that all groups with willingness larger than group K^{pp} can have positive demands. Otherwise, we can construct another partition leading to a larger revenue (please refer to the proof of Lemma 2), or equivalently leading to a less objective value of Level-1′ Problem. This leads to a contradiction. ■

Level-1′ Problem is still a combinatorial optimization problem with a large feasible set of $\boldsymbol{a}$ (similar as the original Level-1). The following result can help us to reduce the size of the feasible set.

Theorem 4: For any effective market size K^{pp} and number of prices J, an optimal partition of PP Problem involves consecutive group indices within clusters.

The proof of Theorem 4 is given in Appendix C. We first prove this result is true for Level-1′ Problem without constraint (28), and further show that this result will not affected by (28). The intuition is that high willingness to pay users should be allocated positive resources with priority. It implies that groups with similar willingness to pays should be partitioned in the same cluster, instead of in several far away clusters. Or equivalently, the group indices within each cluster should be consecutive.

We define $\mathcal{A}$ as the set of all partitions with consecutive group indices within each cluster, and $v(\boldsymbol{a}) = \sum_{j \in \mathcal{J}} N^j \sqrt{\theta^j}$ is the value of objective of Level-1′ Problem for a partition $\boldsymbol{a}$. Algorithm 3 finds the optimal solution of Level-1′. The main idea for this algorithm is to enumerate every possible partition in set $\mathcal{A}$, and then check whether the threshold condition (28) can be satisfied. The main part of this algorithm is to enumerate all partitions in set $\mathcal{A}$ of $C(K^{pp}-1, J-1)$ feasible partitions. Thus the complexity of Algorithm 3 is no more than $\mathcal{O}((K^{pp})^{J-1})$.

2) Search the optimal effective market threshold K^{pp}: We know the optimal market threshold K^{pp} is upper-bounded, i.e., $K^{pp} \leq K^{cp} \leq I$. Thus we can first run Algorithm 1 to calculate the effective market size for the CP scheme K^{cp}. Then, we search the optimal K^{pp} iteratively using Algorithm 3 as an inner loop. We start by letting $K^{pp} = K^{cp}$ and run Algorithm 3. If there is no solution, we decrease K^{pp} by one and run Algorithm 3 again. The algorithm will terminate once we find an effective market threshold where Algorithm 3 has an optimal solution. Once the optimal threshold and the

Algorithm 3 Solve the Level-1$'$ Problem with fixed K^{pp}

function LEVEL-1(K^{pp}, J)
  $k \leftarrow K^{pp}$
  $v^* \leftarrow \sqrt{\sum_{i=1}^{k} N_i\theta_i}$, $\boldsymbol{a}^* = \boldsymbol{0}$
  for $\boldsymbol{a} \in \mathcal{A}$ **do**
    if $\theta_k > \sqrt{\theta^J(\boldsymbol{a})\lambda(\boldsymbol{a})}$ **then**
      if $v(\boldsymbol{a}) < v^*$ **then**
        $v^* \leftarrow v(\boldsymbol{a})$, $\boldsymbol{a}^* \leftarrow \boldsymbol{a}$
      end if
    end if
  end for
  return $\boldsymbol{a}^*$
end function

partition of the clusters are determined, we can further run Algorithm 1 to solve the joint optimal resource allocation and pricing scheme. The pseudo code is given in Algorithm 4 as follows.

Algorithm 4 Solve Partial Price Differentiation Problem

$p_i \leftarrow \theta_i$
$k \Leftarrow$ CP($\{N_i, \theta_i\}_{i\in\mathcal{I}}$, S)_1, $\boldsymbol{a}^* \Leftarrow$ Level-1(k,J)
while $\boldsymbol{a}^* == \boldsymbol{0}$ **do**
  $k \leftarrow k-1$, $\boldsymbol{a}^* \Leftarrow$ Level-1(k,J)
end while
for $j \leftarrow 1, J$ **do**
  $N^j \leftarrow \sum_{i=1}^{k} N_i a_i^j$, $\theta^j \leftarrow \sum_{i=1}^{k} \frac{N_i a_i^j}{N^j}\theta_i$
end for
$\lambda \Leftarrow$ CP($\{N^j, \theta^j\}_{i\in\mathcal{J}}$, S)_2
for $i \leftarrow 1, k$ **do**
  $p_i \leftarrow \sum_{j=1}^{J} a_i^j \sqrt{\theta^j\lambda}$
end for
return $\{p_i\}_{i\in\mathcal{I}}$

In Algorithm 4, it invokes two functions: CP($\{N_i\theta_i\}_{i\in\mathcal{I}}$, S) as described in Algorithm 1 and and Level-1(k, J) as in Algorithm 3. CP($\{N_i\theta_i\}_{i\in\mathcal{I}}$, S) returns a vector with two elements: CP($\{N_i\{\theta_i\}_{i\in\mathcal{I}}$, S)_1 denotes the first element K^{cp}, and CP($\{N_i\theta_i\}_{i\in\mathcal{I}}$, S)_2 denotes the second element λ^* in CP Problem.

The above analysis leads to the following theorem:

Theorem 5: The solution obtained by Algorithm 4 is optimal for PP Problem.

Proof: It is clear that Algorithm 4 enumerates every possible value of the effective market size for PP Problem K^{pp}, and for a given K^{pp} its inner loop Algorithm 3 enumerates every possible partition in set $\mathcal{A}$. Therefore, the result in Theorem 4 follows. ■

Next we discuss the complexity of Algorithm 4. The complexity of Algorithm 1 is $\mathcal{O}(I)$, and we run it twice in Algorithm 4. The worst case complexity of Algorithm 3 is $\mathcal{O}(I^{J-1})$, and we run it no more than $I-J$ times. Thus the whole complexity of Algorithm 4 is no more than $\mathcal{O}(I^J)$, which is polynomial of I.

VI. PRICE DIFFERENTIATION UNDER INCOMPLETE INFORMATION

In Sections III, IV, and V, we discuss various pricing schemes with different implementational complexity level under complete information, the revenues of which can be viewed as the benchmark of practical pricing designs. In this section, we further study the incomplete information scenario, where the SP does not know the group association of each user. The challenge for pricing in this case is that the SP needs to provide the right incentive so that a group i user does not want to pretend to be a user in a different group. It is clear that the CP scheme in Section III and the PP scheme in Section V cannot be directly applied here. The SP scheme in Section IV is a special case, since it does not require the user-group association information in the first place and thus can be applied in the incomplete information scenario directly. On the other hand, we know that the SP scheme may suffer a considerable revenue loss compared with the CP scheme. Thus it is natural to ask whether it is possible to design an incentive compatible differentiation scheme under incomplete information. In this section, we design a quantity-based price menu to incentivize the users to make the right self-selection and achieve the same maximum revenue of the CP scheme under complete information under proper technical conditions. We name it as the Incentive Compatible Complete Price differentiation ($ICCP$) scheme.

In the $ICCP$ scheme, the SP publishes the quantity-based price menu, which consists of several step functions of resource quantity. Users are allowed to freely choose their quantities. The aim of this price menu is to make the users *self-differentiated*, so that to mimic the same result (the same prices and resource allocations) of the CP scheme under complete information. Based on Theorem 1, there are only K (without confusion, we remove the superscript "cp" to simplify the notation) effective groups of users receiving non-zero resource allocations, thus there are K steps of unit prices, $p_1^* > p_2^* > \cdots > p_K^*$ in the price menu. These prices are exactly the same optimal prices that the SP would charge for K effective groups as in Theorem 1. Note that for the $K+1, \ldots, I$ groups, all the prices in the menu are too high for them, then they will still demand zero resource. The quantity is divided into K intervals by $K-1$ thresholds, $s_{th}^1 > s_{th}^2 > \cdots > s_{th}^{K-1}$. The $ICCP$ scheme can specified as follows:

$$p(s) = \begin{cases} p_1^* & \text{when } s > s_{th}^1 \\ p_2^* & \text{when } s_{th}^1 \geq s > s_{th}^2 \\ \vdots & \\ p_K^* & \text{when } s_{th}^{K-1} \geq s > 0. \end{cases} \tag{29}$$

A four-group example is shown in Fig. 8.

Note that in contract to the usual "volume discount", here the price is non-decreasing in quantity. This is motivated by the resource allocation in Theorem 1, that a user with a higher θ_i is charged a higher price for a larger resource allocation. Thus the observable quantity can be viewed as an indication of the unobservable users' willingness to pay, and help to realize price differentiation under incomplete information.

The key challenge in the $ICCP$ scheme is to properly set the quantity thresholds so that users are perfectly segmented

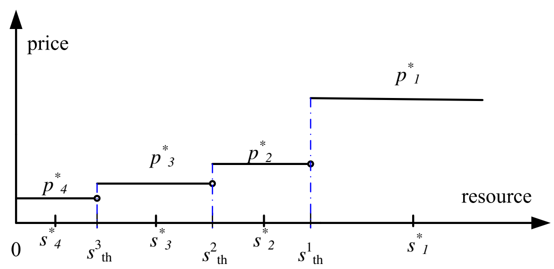

Fig. 8. A four-group example of the $ICCP$ scheme: where the prices $p_1^* > p_2^* > p_3^* > p_4^*$ are the same as the CP scheme. To mimic the same resource allocation as under the CP scheme, one necessary (but not sufficient) condition is $s_{th}^{j-1} \geq s_j^*$ for all j, where s_j^* is the optimal resource allocation of the CP scheme.

through self-differentiation. This is, however, not always possible. Next we derive the necessary and sufficient conditions to guarantee the perfect segmentation.

Let us first study the self-selection problem between two groups: group i and group q with $i < q$. Later on we will generalize the results to multiple groups. Here group i has a higher willingness to pay, but will be charged with a higher price p_i^* in the CP case. The incentive compatible constraint is that a high willingness to pay user can not get more surplus by pretending to be a low willingness to pay user, i.e., $\max_s U_i(s; p_i^*) \geq \max_s U_i(s; p_q^*)$, where $U_i(s;p) = \theta_i \ln(1+s) - ps$ is the surplus of a group i user when it is charged with price p.

Without confusion, we still use s_i^* to denote the optimal resource allocation under the optimal prices in Theorem 1, i.e., $s_i^* = \arg\max_{s_i \geq 0} U_i(s_i; p_i^*)$. We define $s_{i\to q}$ as the quantity satisfying

$$\begin{cases} U_i(s_{i\to q}; p_q^*) = U_i(s_i^*; p_i^*) \\ s_{i\to q} < s_i^* \end{cases} . \tag{30}$$

In other words, when a group i user is charged with a lower price p_q^* and demands resource quantity at $s_{i\to q}$, it achieves the same as the maximum surplus under the optimal price of the CP scheme p_i^*, as showed in Fig. 9. Since the there two solutions of the first equation of (30), we constraint $s_{i\to q}$ to be the one that is smaller than s_i^*.

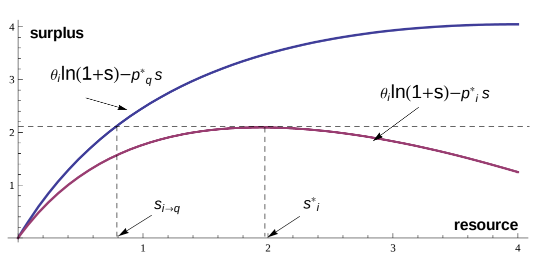

Fig. 9. When the threshold $s_{th}^{q-1} < s_{i\to q}$, the group i user can not obtain $U(s_i^*, p_i^*)$ if it chooses the lower price p_q at a quantity less than s_{th}^{q-1}. Therefore it will automatically choose the high price p_i^* to maximize its surplus.

To maintain the group i users' incentive to choose the higher price p_i^* instead of p_q^*, we must have $s_{th}^{q-1} \leq s_{i\to q}$, which means a group i user can not obtain $U_i(s_i^*, p_i^*)$ if it chooses a quantity less than s_{th}^{q-1}. In other words, it will automatically choose the higher (and the desirable) price p_i^* to maximize its surplus. On the other hand, we must have $s_{th}^{q-1} \geq s_q^*$ in order to maintain the optimal resource allocation and allow a group q user to choose the right quantity-price combination (illustrated in Fig. 8).

Therefore, it is clear that the *necessary and sufficient* condition that the $ICCP$ scheme under incomplete information achieves the same maximum revenue of the CP scheme under complete information is

$$s_q^* \leq s_{i\to q}, \; \forall\, i < q, \forall\, q \in \{2, \ldots, K\}. \tag{31}$$

By solving these inequalities, we can obtain the following theorem (detailed proof in Appendix-D).

Theorem 6: There exist unique thresholds $\{t_1, \ldots, t_{K-1}\}$, such that the $ICCP$ scheme achieves the same maximum revenue as in the complete information case if

$$\sqrt{\frac{\theta_q}{\theta_{q+1}}} \geq t_q \quad \text{for } q = 1, \ldots, K-1.$$

Moreover, t_q is the unique solution of the equation

$$t^2 \ln t - (t^2 - 1) + \frac{t \sum_{k=1}^{q} N_k + N_{q+1}}{S + \sum_{k=1}^{K^{cp}} N_k}(t-1) = 0$$

over the domain $t > 1$.

We want to mention that the condition in Theorem 6 is necessary and sufficient for the case of $K = 2$ effective groups[4]. For $K > 2$, Theorem 6 is sufficient but not necessary. The intuition of Theorem 6 is that users need to be sufficiently different to achieve the maximum revenue.

The following result immediately follows Theorem 6.

Corollary 1: The t_qs in Theorem 6 satisfy $t_q < t_{root}$ for $q = 1, \ldots, K-1$, where $t_{root} \approx 2.21846$ is the larger root of equation $t^2 \ln t - (t^2 - 1) = 0$.

The Corollary 1 means that the users do not need to be extremely different to achieve the maximum revenue.

When the conditions in Theorem 6 are not satisfied, there may be revenue loss by using the pricing menu in (29). Since it is difficult to explicitly solve the parameterized transcend equation (30), we are not able to characterize the loss in a closed form yet.

We believe that the above analysis method can be extended to the case of partial price differentiation under incomplete information, since PP Problem can be viewed as CP Problem for all effective super-groups. Once the super-group partition can be solved by the searching Algorithm 4, then for any given realization of parameters, we can numerically calculate and determine whether $ICCP$ scheme (for super-groups) is achievable under this realization of parameters. However, since there is no analytical solutions for the super-group (or cluster) partition problem, it is also not easy to characterize the analytical bound as Theorem 6 for an incentive compatible partial price different scheme. We will leave it for a further study.

VII. Numerical Results

We provide numerical examples to quantitatively study several key properties of price differentiation strategies in this section.

[4]There might be other groups who are not allocated positive resource under the optimal pricing.

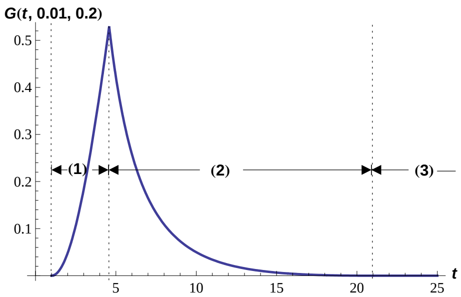

Fig. 10. One example of the revenue gain $G(t, 0.01, 0.2)$ for the CP scheme. It is clear that the revenue gain can be divided into three regions. Region(1), increasing region, where $K^{cp} = K^{sp} = 2$, and the revenue gain comes from the differentiation gain. Region(2), decreasing region, where $K^{cp} = 2$, $K^{sp} = 1$, and the revenue gain comes from larger effective market and differentiation gain. Region(3), zero region, where $K^{cp} = K^{sp} = 1$, and is a degenerating case where two pricing scheme coincide.

A. When is price differentiation most beneficial?

Definition 1: (Revenue gain) We define the revenue gain G of one pricing scheme as the ratio of the revenue difference (between this pricing scheme and the single pricing scheme) normalized by the revenue of single pricing scheme.

In this subsection, we will study the revenue gain of the CP scheme, i.e., $G(\boldsymbol{N}, \boldsymbol{\theta}, S) \triangleq \frac{R_{cp}-R_{sp}}{R_{sp}}$, where $\boldsymbol{N} \triangleq \{N_i, \forall i \in \mathcal{I}\}$ denotes the number of users in each groups, $\boldsymbol{\theta} \triangleq \{\theta_i, \forall i \in \mathcal{I}\}$ denotes their willingness to pays, and S is the total resource. Notice that this gain is the maximum possible differentiation gain among all PP schemes.

We first study a simple two-group case. According to Theorems 1 and 2, the revenue under the SP scheme and the CP scheme can be calculated as follows:

$$R^{sp} = \begin{cases} \frac{S(N_1\theta_1+N_2\theta_2)}{N_1+N_2+S} & 1 \le t < \sqrt{\frac{S+N_1}{N_1}}, \\ \frac{SN_1\theta_1}{N_1+S} & t \ge \sqrt{\frac{S+N_1}{N_1}}, \end{cases}$$

and

$$R^{cp} = \begin{cases} \frac{S(N_1\theta_1+N_2\theta_2)+N_1N_2(\sqrt{\theta_1}-\sqrt{\theta_2})^2}{N_1+N_2+S} & 1 \le t < \frac{S+N_1}{N_1}, \\ \frac{SN_1\theta_1}{N_1+S} & t \ge \frac{S+N_1}{N_1}. \end{cases}$$

where $t = \sqrt{\frac{\theta_1}{\theta_2}} > 1$.

The revenue gain will depend on five parameters, S, N_1, θ_1, N_2 and θ_2. To simplify notations, let $N = N_1 + N_2$ be the total number of the users, $\alpha = \frac{N_1}{N}$ the percentage of group 1 users, and $\bar{s} = \frac{S}{N}$ the level of normalized available resource. Thus the revenue gain can be expressed as

$$G(t, \alpha, \bar{s}) = \begin{cases} \frac{\alpha(1-\alpha)(t-1)^2}{\bar{s}(1+\alpha(t^2-1))} & 1 < t < \sqrt{\frac{\bar{s}+\alpha}{\alpha}}, \\ \frac{(1-\alpha)(\bar{s}+\alpha-t\alpha)^2}{\alpha\bar{s}(1+\bar{s})t^2} & \sqrt{\frac{\bar{s}+\alpha}{\alpha}} \le t \le \frac{\bar{s}+\alpha}{\alpha}. \end{cases} \quad (32)$$

Next we discuss the impact of each parameter.

Observation 1: In terms of the parameter t, G monotonically increases in $(1, \sqrt{\frac{\bar{s}+\alpha}{\alpha}})$ and decrease in $[\sqrt{\frac{k+\alpha}{\alpha}}, \frac{k+\alpha}{\alpha})$. The maximum is obtained at $t_{G-max} = \sqrt{\frac{k+\alpha}{\alpha}}$, when the resource allocated to the group 2 user just becomes zero in the SP scheme.

One example is showed in Fig.10.

It is clear that the revenue gain is not monotonic in the willingness to pay ratio. Its behavior can be divided into three regions: the increasing Region (1) with $t \in (1, \sqrt{\frac{\bar{s}+\alpha}{\alpha}})$, the decreasing Region (2) with $t \in [\sqrt{\frac{k+\alpha}{\alpha}}, \frac{k+\alpha}{\alpha})$, and the zero Region (3) with $t \ge \frac{k+\alpha}{\alpha}$.

It is also interesting to note that three regions are closed related to the effective market sizes: $K^{sp} = K^{cp} = 2$ in Region (1); $K^{sp} = 1$ and $K^{cp} = 2$ in Region (2); and $K^{cp} = K^{sp} = 1$ in Region (3) where the CP scheme degenerates to the SP scheme. The peak point of the revenue gain correspond to the place where the effective market of SP Scheme changes.

Intuitively, the CP scheme increases the revenue by charging the high willingness groups with high prices, thus the revenue gain increases first when the difference of willingness to pays increase. However, when the difference of willingness to pay is very large, the CP scheme obtain most revenue from the high willingness to pay users, while the SP scheme declines the low willingness to pay users but serves the high willingness to pays only. Both schemes lead to similar resource allocation in this region, and thus the revenue gain decreases as the difference of willingness to pays increases.

Figure 10 shows the revenue gain under usage-based pricing can be very high in some scenario, e.g., over 50% in this example. We can define this peak revenue gain as

$$G_{max}(\alpha, \bar{s}) = \max_{t \ge 1} G(t, \alpha, \bar{s}) = \frac{(\alpha-1)(\sqrt{\bar{s}+\alpha}-\sqrt{\alpha})^2}{\bar{s}(1+\bar{s})}.$$

Figure 11 is shown how G_{max} changes in $\bar{s}$ with different parameters α.

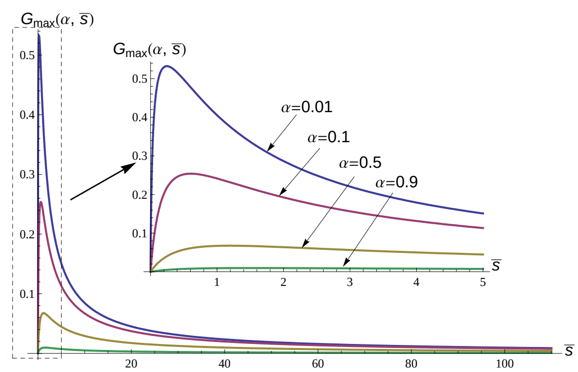

Fig. 11. For a fixed $\bar{s}$, $G_{max}(\alpha, \bar{s})$ monotonically increases in α. For a fixed α, $G_{max}(\alpha, \bar{s})$ first increases in $\bar{s}$, and then decreases in $\bar{s}$

Observation 2: For a fixed $\bar{s}$, $G_{max}(\alpha, \bar{s})$ monotonically decreases in α .

When α is small, which means high willingness to pay users are minorities in the effective market, the advantage of price differentiation is very evident. As shown in Fig. 11, when $\alpha = 0.1$, the maximum possible revenue gain can be over than 20%; and when $\alpha = 0.01$, this gain can be even higher than 50%. However, when high willingness to pay users are majority, the price differentiation gain is very limited, for example, the gain is no larger than 8% and 2% for $\alpha = 0.5$ and 0.9, respectively.

Intuitively, high willingness to pay users are the most profitable users in the market. Ignoring them is detrimental in terms of revenue even if they only occupy a small fraction of the population. Since the SP scheme is set based on the average willingness to pay of the effective market, the high willingness to pay users will be ignored (in the sense of not

charging the desirable high price) when α is small. In contrast, ignoring the low willingness to pay users when α is large is not a big issue.

Observation 3: For parameter k, $G_{max}(\alpha, \bar{s})$ is not a monotonic function in $\bar{s}$. Its shape looks like a skewed bell. The gain is either small when $\bar{s}$ is very small or very large.

Small $\bar{s}$ means that resource is very limited, and both schemes allocates the resource to high willingness to pay users (see the discussion of the threshold structure in Sections III and IV), and thus there is not much difference between two pricing schemes. While $\bar{s}$ is very large, i.e., the resource is abundant, the prices and the resource allocation with or without differentiation become similar (which can be easily checked from formulations in Theorems 1 and 2). In these two scenarios, similar resource allocations lead to similar revenues. These explains the bell shape for parameter $\bar{s}$.

Based on the above observations, we find that the revenue gain can be very high under two conditions. First, the high willingness to pay users are minorities in the effective market. Second, the total resource is comparatively limited.

For cases with three or more groups, the analytical study becomes much more challenging due to many more parameters. Moreover, the complex threshold structure of the effective market makes the problem even complicated. We will present some numerical studies to illustrate some interesting insights.

For illustration convenience, we choose a three-group example and three different sets of parameters as shown in Table II. To limit the dimension of the problem, we set the parameters such that the total number of users and the average willingness to pay (i.e., $\bar{\theta} = \sum_{i=1}^{3} N_i\theta_i / (\sum_{i=1}^{3} N_i)$) of all users are the same across three different parameter settings. This ensures that the SP scheme achieves the same revenue in three different cases when resource is abundant. Figure 12 illustrates how the differentiation gain changing changes in resource S.

TABLE II
PARAMETER SETTINGS OF A THREE-GROUP EXAMPLE

	θ_1	N_1	θ_2	N_2	θ_3	N_3	$\bar{\theta}$
Case 1	9	10	3	10	1	80	2
Case 2	3	33	2	33	1	34	2
Case 3	2.2	80	1.5	10	1	10	2

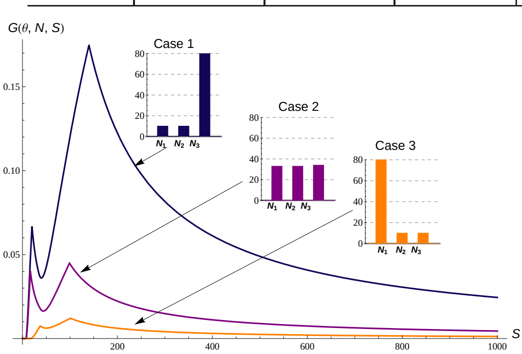

Fig. 12. An example of the revenue gain of the three-group market with the same average willingness to pay

Similar as the analytical study of the two-group case, Fig. 12 shows that the revenue gain is large only when the high willingness to pay users are minorities (e.g. case 1) in the effective market and the resource is limited but not too small ($100 \leq S \leq 150$ in all three cases). When resource S is large enough (e.g., ≥ 150), the gain will gradually diminish to zero as the resource increases. For each curve in Fig. 12, there are two peak points. Each peak point represents a change of the effective market threshold in the SP scheme, i.e., when the resource allocation to a group becomes zero. In numerical studies of networks with $I > 3$ groups (not shown in this paper), we have observed the similar conditions for achieving a large differentiation gain and the phenomenon of $I-1$ peak points.

B. What is the best tradeoff of Partial Price Differentiation?

In Section V, we design Algorithm 4 that optimally solves PP Problem with a polynomial complexity. Here we study the tradeoff between total revenue and the implementational complexity.

To illustrate the tradeoff, we consider a five-group example with parameters shown in Table III. Note that high willingness to pay users are minorities here. Figure 13 shows the revenue gain G as a function of total resource S under different PP schemes (including CP scheme as a special case), and Fig. 14 shows how the effective market thresholds change with the total resource.

TABLE III
PARAMETER SETTING OF A FIVE-GROUP EXAMPLE

group index i	1	2	3	4	5
θ_i	16	8	4	2	1
N_i	2	3	5	10	80

We enlarge Fig. 13 and Fig. 14 within the range of $S \in [0, 50]$, which is the most complex and interesting part due to several peak points. Similar as Fig. 12, we observe $I - 1 = 4$ peak points for each curve in Fig. 13. Each peak point again represents a change of effective market threshold of the single pricing scheme, as we can easily verify by comparing Fig. 14 with Fig. 13.

As the resource S increases from 0, all gains in Fig. 13 first overlap with each other, then the two-price scheme (blue curve) separates from the others at $S = 3.41$, after that the three-price scheme (purple curve) separates at $S = 8.89$, and finally the four-price scheme (dark yellow curve) separates at near $S = 20.84$. These phenomena are due to the threshold structure of the PP scheme. When the resource is very limited, the effective markets under all pricing scheme include only one group with the highest willingness to pay, and all pricing schemes coincide with the SP scheme. As the resource increases, the effective market enlarges from two groups to finally five groups. The change of the effective market threshold can be directly observed in Fig. 14. Comparing across different curves in Fig. 14, we find that the effective market size is non-decreasing with the number of prices for the same resource S. This agrees with our intuition in Section IV-B, which states that the size of effective market indicates the degree of differentiation.

Figure 13 provides the service provider a global picture of choosing the most proper pricing scheme according to achieve the desirable financial target under a certain parameter setting.

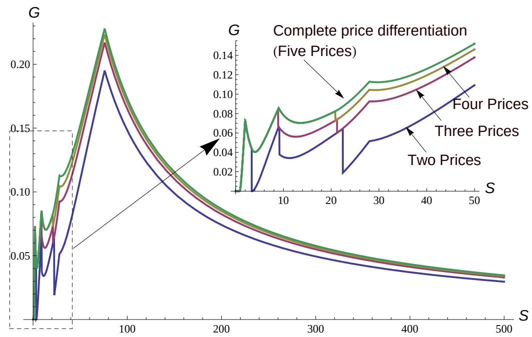

Fig. 13. Revenue gain of a five-group example under different price differentiation schemes

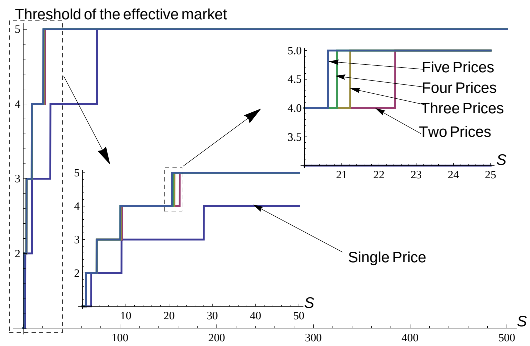

Fig. 14. Corresponding thresholds of effective markets of Fig. 13's example

For example, if the total resource $S = 100$, the two-price scheme seems to be a sweet spot, as it achieves a differential gain of 14.8% comparing to the SP scheme and is only 2.4% worse than the CP scheme with five prices.

VIII. Conclusion

In this paper, we study the revenue-maximizing problem for a monopolistic service provider under both complete and incomplete network information.

The total revenue and the implementational complexity are two important concerns when the service provider determines the pricing plan. To investigate the tradeoff between these two factors, we propose three pricing schemes in the complete network information scenario: the complete price differentiation scheme, the single pricing scheme (no price differentiation), and the partial price differentiation scheme. The partial price differentiation is the most general formulation and includes the other two schemes as its special cases. Solving the partial price differentiation problem optimally is challenging, since it involves combinatorial optimization of customer partitioning. However, by utilizing the unique structure of the problem, we designed a polynomial-time algorithm to optimally solve it, and quantize the trade-off between implementational complexity and total revenue with numerical results. The obtained results can be a good reference for making practical pricing plans.

In the incomplete information scenario, designing an incentive-compatible differentiation pricing scheme is rather difficult. However, we show one possible way to achieve this by providing a quantity-based pricing scheme, where a user needs to pay a higher unit price with a larger quantity purchase. We characterize the necessary and sufficient condition under which such scheme achieves the same optimal revenue as under complete information.

Some topics deserve further investigations, including the design of partial price differentiation under general utility functions and the incomplete information network.

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Appendix

A. Proof of Proposition 1

Proof: We first focus on the key water-filling problems that we solve for the two pricing schemes (the CP scheme on the LHS and the SP scheme on the RHS):

$$\sum_{i\in\mathcal{I}} N_i \left(\sqrt{\frac{\theta_i}{\lambda^*}}-1\right)^+ = S = \sum_{i\in\mathcal{I}} N_i \left(\frac{\theta_i}{p^*}-1\right)^+ \quad (33)$$

Let $\theta = \frac{p^{*2}}{\lambda^*}$ be the solution of the equation of $\sqrt{\frac{\theta}{\lambda^*}} = \frac{\theta}{p^*}$. By comparing it with θ_i, $i \in \mathcal{I}$, there are three cases:

- Case 1: $\theta > \theta_1 \Rightarrow \sqrt{\frac{\theta_i}{\lambda^*}} = \frac{\sqrt{\theta_i}\sqrt{\theta}}{p^*} > \frac{\theta_i}{p^*}, \forall i \in \mathcal{I}$. This case can not be possible. Since if every term in the left summation is strictly larger than its counterpart in the right summation, then (33) can not hold.
- Case 2: $\theta_I \geq \theta \Rightarrow \sqrt{\frac{\theta_i}{\lambda^*}} = \frac{\sqrt{\theta_i}\sqrt{\theta}}{p^*} \leq \frac{\theta_i}{p^*}, \forall i \in \mathcal{I}$. Similarly as Case 1, it can not hold, either.
- Case 3: $\exists k, \ s.t.\ 1 \leq k < I \text{ and } \theta_k \geq \theta \geq \theta_{k+1}$

$$\Rightarrow \begin{cases} \sqrt{\frac{\theta_i}{\lambda^*}} = \underbrace{\frac{\sqrt{\theta_i}\sqrt{\theta}}{p^*} \leq \frac{\theta_i}{p^*}, i = 1, 2, \dots, k;}_{\text{The equality holds only when } \theta=\theta_k \text{ and } i=k.} \\ \sqrt{\frac{\theta_i}{\lambda^*}} = \underbrace{\frac{\sqrt{\theta_i}\sqrt{\theta}}{p^*} \geq \frac{\theta_i}{p^*}, \ i = k+1, \dots, I.}_{\text{The equality holds only when } \theta=\theta_{k+1} \text{ and } i=k+1.} \end{cases}$$

Similar argument as the above two case, we have $K^{cp} \geq k$ and $K^{sp} \geq k$, otherwise (33) can not hold. Further, $K^{cp} \geq K^{sp}$, since if $\frac{\theta_{K^{sp}}}{p^*} - 1 > 0$, then $\sqrt{\frac{\theta_{K^{cp}}}{\lambda^*}} - 1 > 0$.

By Theorems 1 and 2, we prove the proposition. ■

B. Proof of Lemma 2

Proof: We can first prove the following lemma.

Lemma 3: (See [21] for proof details.) Suppose an effective market of the single pricing scheme is denoted as $\mathcal{K} = \{1, 2, \dots, K\}$. If we add a new group v of N_v users with $\theta_v > \theta_K$, then the revenue strictly increases.

Now Let us prove Lemma 2 by contradiction. Suppose that the group indices of the effective market under the optimal partition $\boldsymbol{a}$ is not consecutive. Suppose that group i is not an effective group, and there exists some group j, $j > i$, which is an effective group. We consider a new partition $\boldsymbol{a}'$ by putting group i into the cluster to which group j belongs, and keeping other groups unchanged. According to Lemma 3, the revenue under partition $\boldsymbol{a}'$ is greater than that under partition $\boldsymbol{a}$, thus partition $\boldsymbol{a}$ is not optimal. This contradicts to our assumption and thus completes the proof. ■

C. Proof of Theorem 4

Proof: For convenience, we we use the notation $(\cdots \cup \cdots | \cdots \cup \cdots | \cdots)$ to denote a partition with the groups between bars connected with "∪" representing a cluster, e.g., three partitions for $J = 2, K^{pp} = 3$ are $(1|2\cup3)$, $(1\cup2\,|\,3)$ and $(1\cup3\,|\,2)$. In addition, we introduce the *compound group* to simplify the notation of complex clusters with multiple groups. A cluster containing group i can be simply represented as $Pre(i) \cup i \cup Post(i)$, where $Pre(i)$ (or $Post(i)$) refers as a compound group composing of all the groups with willingness to pay larger (or smaller) than that of group i in the cluster. Note that the compound groups can be empty in certain cases.

We prove Theorem 4 by Contraction. For any partition with discontinuous group indices, we can construct another partition with a better revenue.

First, how can we know that a partition has discontinues group indices? We can simply check the indices continuity for every single group, from group 1 to group K^{pp}. Without loss of generality, suppose that the group within each cluster is arranged according to an increasing indices order. Thus, for a group c within a cluster $\mathcal{C}$, suppose that it has a next neighbor denoted as group d in $\mathcal{C}$. Then the group indices until c are consecutive if and only if $d - c = 1$. If group c has no next neighbor, then the group indices until c are consecutive if and only if group $c + 1$ has no previous neighbor in the cluster that group $c + 1$ belongs to.

Lemma 4: (See [21] for proof details.) For a four-group effective market with two prices, i.e., $K^{pp} = 4$, $J = 2$, any optimal partition involves consecutive group indices within clusters.

Now let us prove the general case in Theorem 4. Suppose we check the group continuity for a partition with discontinues group indices. We do not find any gap until group u_1 in cluster $\mathcal{U}$. We denote group u_1 next neighbor in cluster $\mathcal{U}$ is group u_2. Since there is a gap between u_1 and u_2, there exists a group v in another cluster $\mathcal{V}$ and satisfying $v = u_1 + 1 < u_2$. We can view $\mathcal{U}$ as $(Pre(u_2) \cup Post(u_1))$, and $\mathcal{V}$ as $(v \cup Post(v))$. By Lemma 4, we can construct a better partition by rearranging the two clusters $\mathcal{U}$ and $\mathcal{V}$, while keeping other clusters unchanged, since this four-group partition does not satisfies consecutive group indices. (See [21] for more construction details.) ■

D. Proof of Theorem 6

Proof: Since $U_i(s, p_q)$ is a strictly increasing function in the interval $[0, s_i^*]$, then (31) holds, if and only if the following inequality holds:

$$U_i(s_q^*, p_q) \leq U_i(s_{i\to q}, p_q), \ \forall i < q. \quad (34)$$

Since $t_{1q} > \cdots > t_{Kq}$, (34) can be simplified to

$$t_{q-1q}^2 \ln t_{q-1q} - (t_{q-1q}^2 - 1) + \frac{\sum_{k=1}^{K} N_k t_{kq}}{\sum_{k=1}^{K} N_k + S}(t_{q-1q} - 1) \geq 0, \quad (35)$$

where $t_{iq} = \sqrt{\frac{\theta_i}{\theta_q}}$. With a slight abuse of notation, we abbreviate t_{q-1q} as t_q, $(q = 2, \dots, K)$ in the sequel. It is easy to see that the following inequality is the necessary and sufficient condition of (35) for $q = 2$, and sufficient condition of (35) for $q > 2$:

$$t_q^2 \ln t_q - (t_q^2 - 1) + \frac{t_q \sum_{k=1}^{q-1} N_k + N_q}{\sum_{k=1}^{K} N_k + S}(t_q - 1) \geq 0. \quad (36)$$

Let $g(t)$ be the left hand side of the inequality (36). It is easy to check that $g(t)$ is a convex function, with $g(1) = 0$, $g(\infty) = \infty$ and $g'(1) < 0$. So there exists a root $t_q > 1$. When $t > t_q$, the inequality (36) holds, thus (34) holds, and the conclusion in Theorem 6 follows. ■